Chapter 12

Equilibrium and Elasticity

HyperPhysics: Equilibrium

Simple Machines

12.2 Equilibrium

- What allows objects to be stable in spite of forces acting on it?
- Under what conditions do objects deform?

The two requirements for the state of **equilibrium** are:

1. The linear momentum of the center of mass is constant.

2. The angular momentum about the center of mass, or about any other point, is also constant.

The balancing rock of Fig. 12-1 is an example of an object that is in ***static equilibrium****. That is, in this situation the constants in the above requirements are zero.*

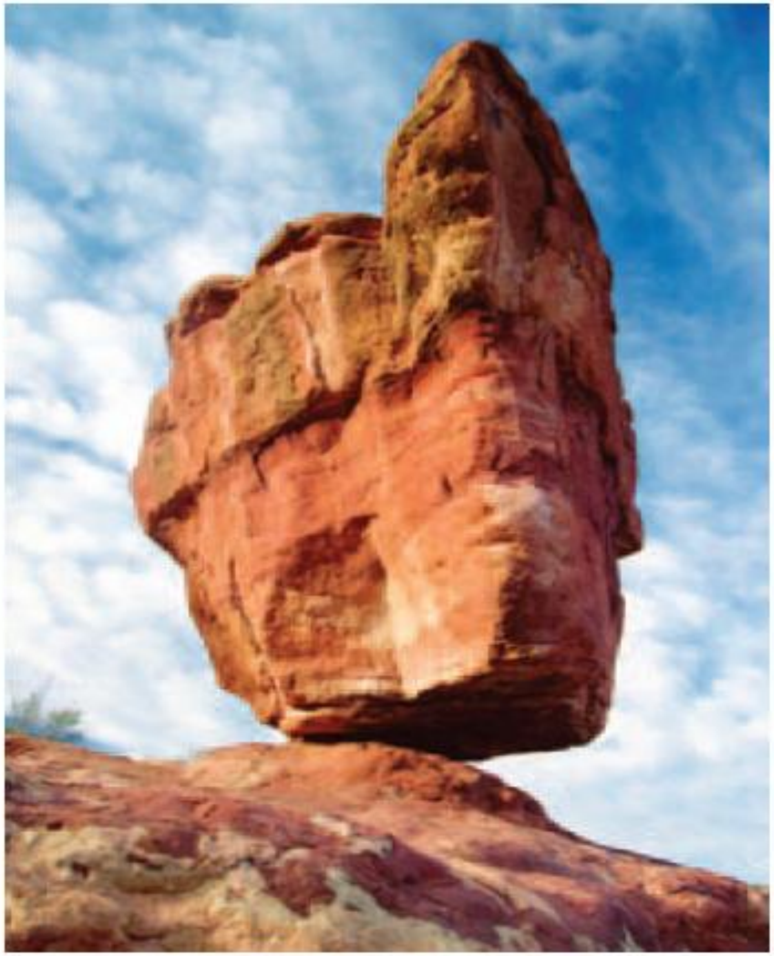


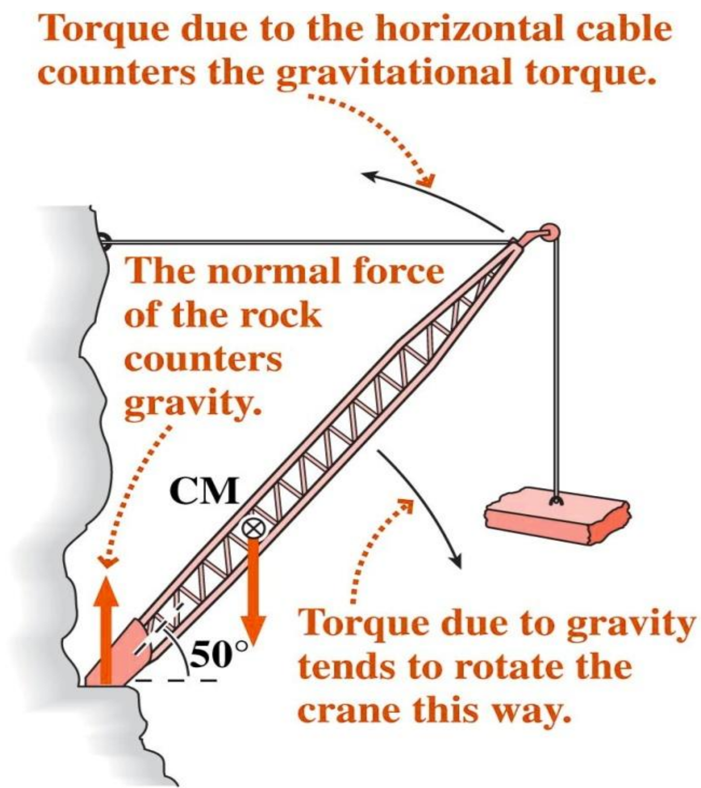
Fig. 12-1 A balancing rock. Although its perch seems precarious, the rock is in static equilibrium. *(Symon Lobsang/Photis/ Jupiter Images Corp.)*

12.3 The Requirements of Equilibrium

1. The vector sum of all the external forces that act on the body must be zero.
2. The vector sum of all external torques that act on the body, measured about *any* possible point, must also be zero.

$\vec{F}_{net} = 0$ (balance of forces). + $\vec{\tau}_{net} = 0$ (balance of torques).

Balance of forces	Balance of torques
$F_{net,x} = 0$	$\tau_{net,x} = 0$
$F_{net,y} = 0$	$\tau_{net,y} = 0$
$F_{net,z} = 0$	$\tau_{net,z} = 0$



Another requirement for static equilibrium:

3. The linear momentum $\vec{P}$ of the body must be zero.

12.4 The Center of Gravity

The gravitational force $\vec{F}_g$ on a body effectively acts at a single point, called the **center of gravity** (cog) of the body.

If $\vec{g}$ is the same for all elements of a body, then the body's center of gravity (cog) is coincident with the body's center of mass (com).

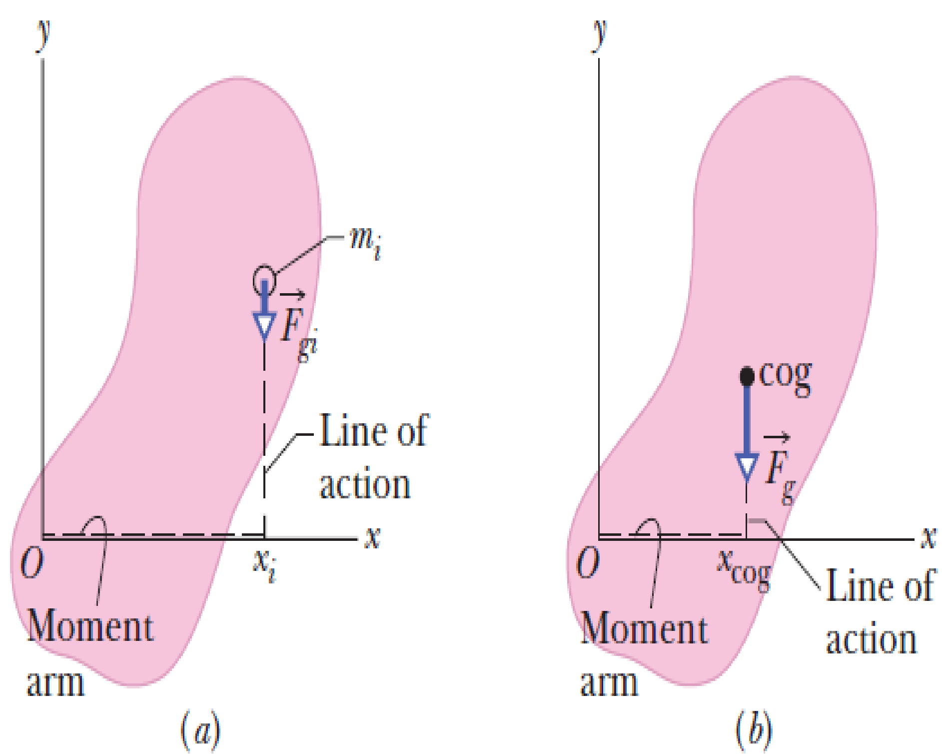
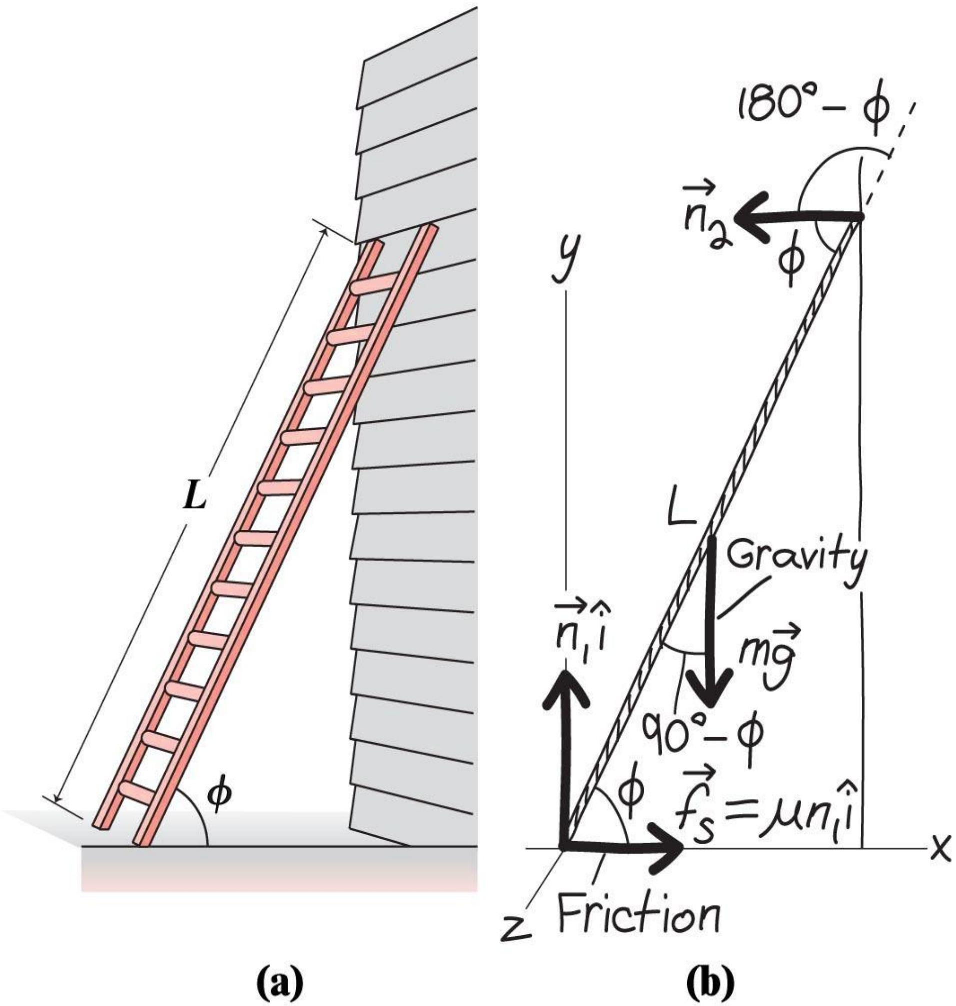


Fig. 12-4 (a) An element of mass m_i in an extended body. The gravitational force $\vec{F}_{gi}$ on the element has moment arm x_i about the origin O of the coordinate system. (b) The gravitational force $\vec{F}_g$ on a body is said to act at the center of gravity (cog) of the body. Here $\vec{F}_g$ has moment arm x_{cog} about origin O.

12.4 Example: Static equilibrium, leaning ladder

- At what angle will the ladder slip?

The ladder and the forces on it:



Forces in both directions sum to 0:

$$\text{Force, } x: \quad \mu n_1 - n_2 = 0$$

$$\text{Force, } y: \quad n_1 - mg = 0$$

The torques are all perpendicular to the plane of the page, so there's only one torque equation:

$$\text{Torque:} \quad L n_2 \sin\phi - \frac{L}{2} mg\cos\phi = 0$$

Solve the 3 equations to get:

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{1}{2\mu}$$

12.4 Example: Static equilibrium

In Fig. 12-5*a*, a ladder of length $L = 12$ m and mass $m = 45$ kg leans against a slick wall (that is, there is no friction between the ladder and the wall). The ladder's upper end is at height $h = 9.3$ m above the pavement on which the lower end is supported (the pavement is not frictionless). The ladder's center of mass is $L/3$ from the lower end, along the length of the ladder. A firefighter of mass $M = 72$ kg climbs the ladder until her center of mass is $L/2$ from the lower end. What then are the magnitudes of the forces on the ladder from the wall and the pavement?

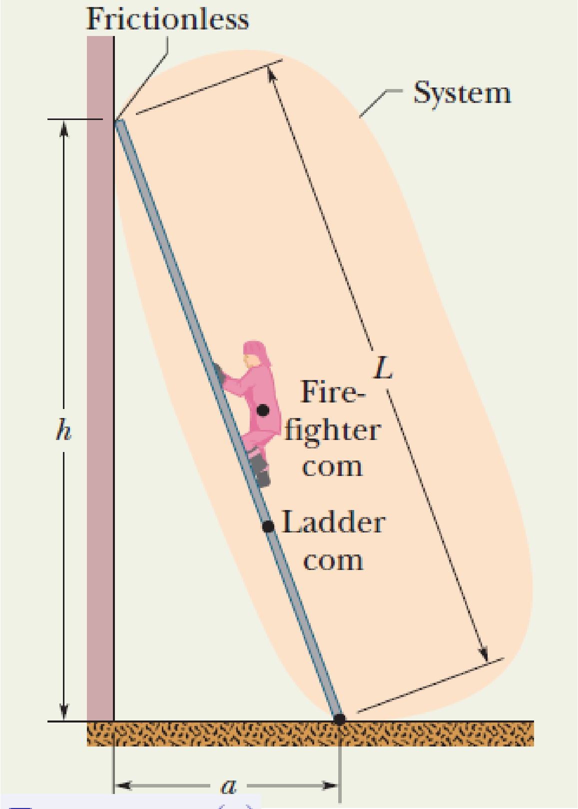


Fig. 12-5 (*a*)

$$\tau_{\text{net},z} = 0$$

$$-(h)(F_w) + (a/2)(Mg) + (a/3)(mg) + (0)(F_{px}) + (0)(F_{py}) = 0. \quad (12\text{-}18)$$

$$a = \sqrt{L^2 - h^2} = 7.58 \text{ m}.$$

$$F_w = \frac{ga(M/2 + m/3)}{h} = \frac{(9.8 \text{ m/s}^2)(7.58 \text{ m})(72/2 \text{ kg} + 45/3 \text{ kg})}{9.3 \text{ m}} = 407 \text{ N} \approx 410 \text{ N}. \quad \text{(Answer)}$$

$$F_{\text{net},x} = 0 \Rightarrow F_w - F_{px} = 0, \quad F_{px} = F_w = 410 \text{ N}.$$

$$F_{\text{net},y} = 0 \Rightarrow F_{py} - Mg - mg = 0,$$

$$F_{py} = (M + m)g = (72 \text{ kg} + 45 \text{ kg})(9.8 \text{ m/s}^2) = 1146.6 \text{ N} \approx 1100 \text{ N}.$$

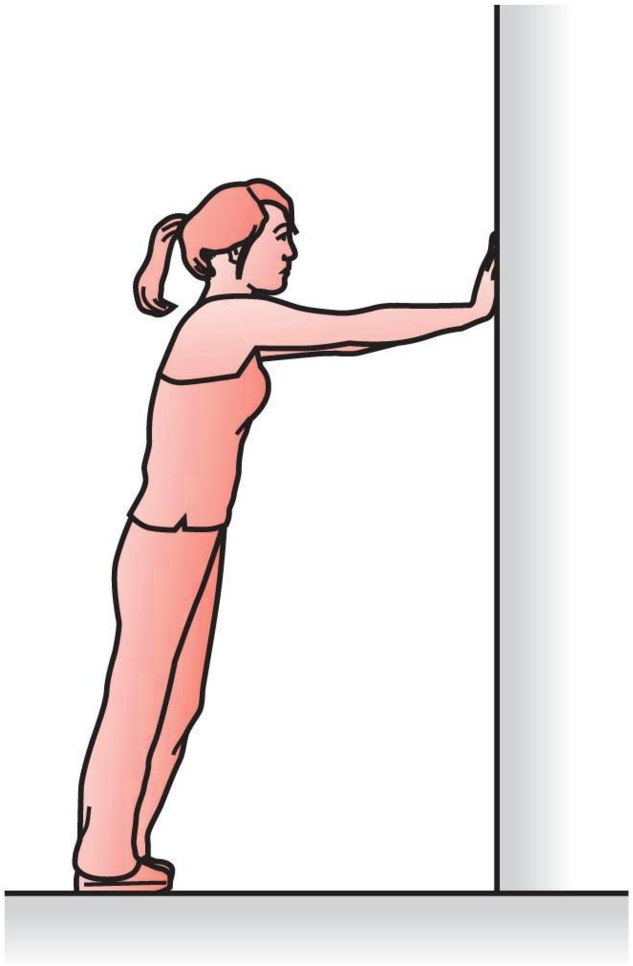
Clicker question

The figure shows a person in static equilibrium leaning against a wall. Which one of the following *must* be true?

A. There must be a frictional force at the wall, but not necessarily at the floor.

B. There must be a frictional force at the floor, but not necessarily at the wall.

C. There must be frictional forces at both wall and floor.



12.4 Example: Static equilibrium

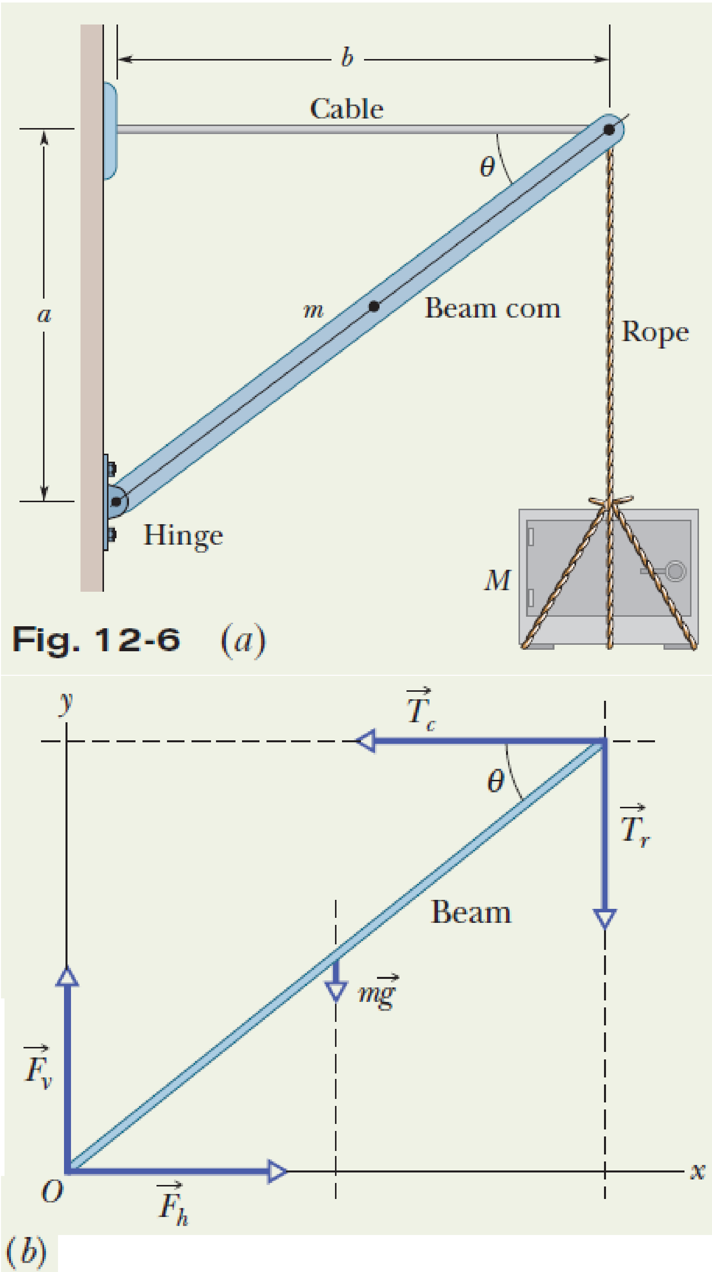


Figure 12-6*a* shows a safe (mass M = 430 kg), hanging by a rope (negligible mass) from a boom (a = 1.9 m and b = 2.5 m) that consists of a uniform hinged beam (m = 85 kg) and horizontal cable (negligible mass).

(a) What is the tension T_c in the cable? In other words, what is the magnitude of the force $\vec{T}_c$ on the beam from the cable?

Calculations: Let us start with Eq. 12-9 ($\tau_{\text{net},z} = 0$). Note that we are asked for the magnitude of force $\vec{T}_c$ and not of forces $\vec{F}_h$ and $\vec{F}_v$ acting at the hinge, at point O. To eliminate $\vec{F}_h$ and $\vec{F}_v$ from the torque calculation, we should calculate torques about an axis that is perpendicular to the figure at point O. Then $\vec{F}_h$ and $\vec{F}_v$ will have moment arms of zero. The lines of action for $\vec{T}_c$, $\vec{T}_r$, and $m\vec{g}$ are dashed in Fig. 12-6*b*. The corresponding moment arms are a, b, and $b/2$.

Writing torques in the form of $r_\perp F$ and using our rule about signs for torques, the balancing equation $\tau_{\text{net},z} = 0$ becomes

$$(a)(T_c) - (b)(T_r) - (\tfrac{1}{2}b)(mg) = 0. \qquad (12\text{-}19)$$

Substituting Mg for T_r and solving for T_c, we find that

$$T_c = \frac{gb(M + \frac{1}{2}m)}{a}$$

$$= \frac{(9.8\ \text{m/s}^2)(2.5\ \text{m})(430\ \text{kg} + 85/2\ \text{kg})}{1.9\ \text{m}}$$

$$= 6093\ \text{N} \approx 6100\ \text{N}. \qquad \text{(Answer)}$$

12.4 Example: Static equilibrium, cont.

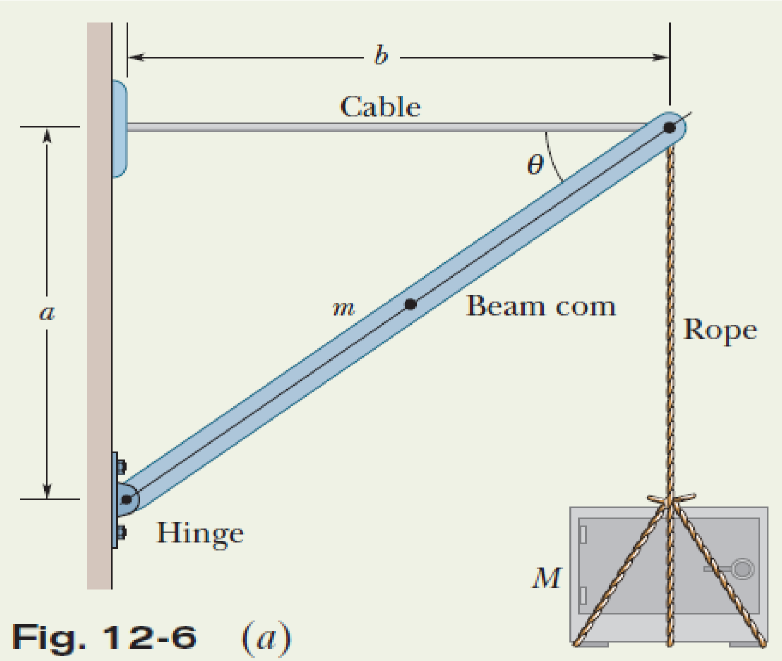
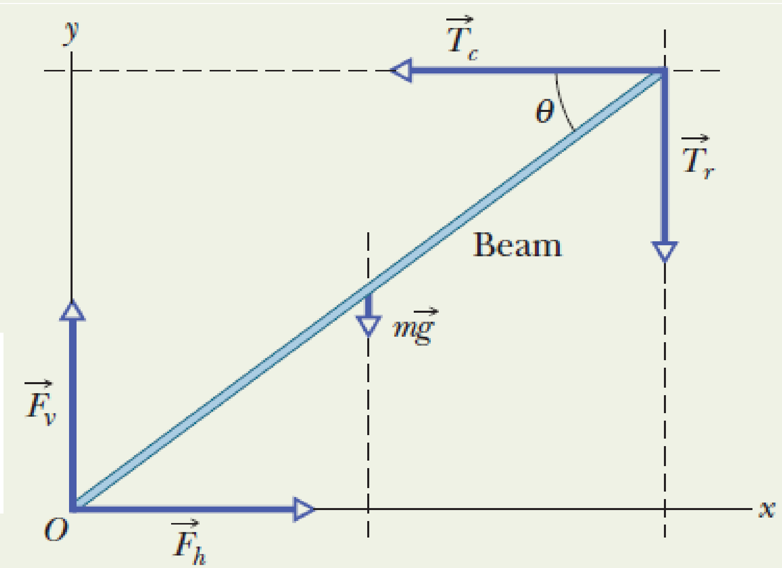


Fig. 12-6 (*a*)



(*b*)

(b) Find the magnitude F of the net force on the beam from the hinge.

Calculations: For the horizontal balance, we write $F_{\text{net},x} = 0$ as

$$F_h - T_c = 0, \qquad (12\text{-}20)$$

and so

$$F_h = T_c = 6093 \text{ N}.$$

For the vertical balance, we write $F_{\text{net},y} = 0$ as

$$F_v - mg - T_r = 0.$$

Substituting Mg for T_r and solving for F_v, we find that

$$F_v = (m + M)g = (85 \text{ kg} + 430 \text{ kg})(9.8 \text{ m/s}^2)$$
$$= 5047 \text{ N}.$$

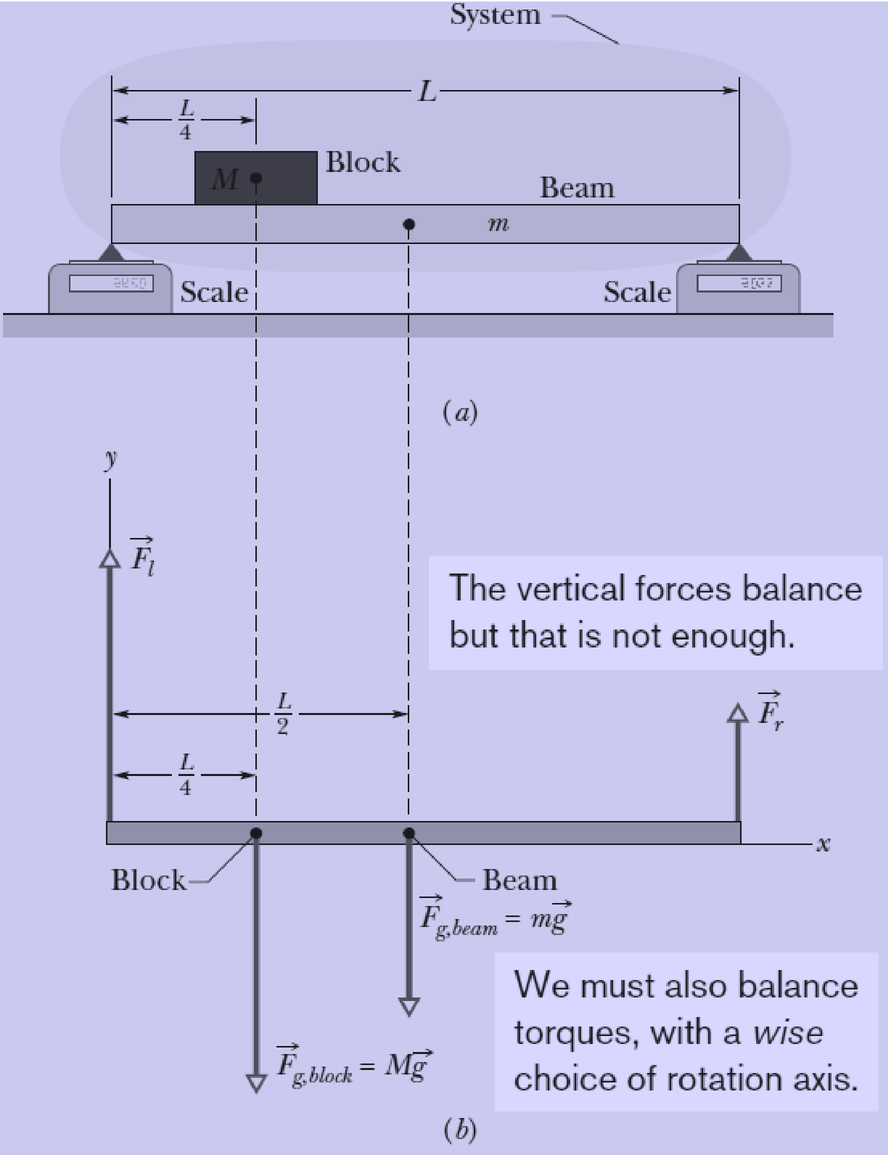
From the Pythagorean theorem, we now have

$$F = \sqrt{F_h^2 + F_v^2}$$
$$= \sqrt{(6093 \text{ N})^2 + (5047 \text{ N})^2} \approx 7900 \text{ N}. \qquad \text{(Answer)}$$

Note that F is substantially greater than either the combined weights of the safe and the beam, 5000 N, or the tension in the horizontal wire, 6100 N.

12.4 Example: Static equilibrium

In Fig. 12-7*a*, a uniform beam, of length L and mass $m = 1.8$ kg, is at rest on two scales. A uniform block, with mass $M = 2.7$ kg, is at rest on the beam, with its center a distance $L/4$ from the beam's left end. What do the scales read?



The vertical forces balance but that is not enough.

We must also balance torques, with a *wise* choice of rotation axis.

$$(F_{net,x} = 0) \rightarrow F_l + F_r - Mg - mg = 0.$$

$$\tau_{net,z} = 0$$

$$(0)(F_l) - (L/4)(Mg) - (L/2)(mg) + (L)(F_r) = 0,$$

$$\begin{aligned} F_r &= \tfrac{1}{4}Mg + \tfrac{1}{2}mg \\ &= \tfrac{1}{4}(2.7\text{ kg})(9.8\text{ m/s}^2) + \tfrac{1}{2}(1.8\text{ kg})(9.8\text{ m/s}^2) \\ &= 15.44\text{ N} \approx 15\text{ N}. \qquad \text{(Answer)} \end{aligned}$$

$$\begin{aligned} F_l &= (M + m)g - F_r \\ &= (2.7\text{ kg} + 1.8\text{ kg})(9.8\text{ m/s}^2) - 15.44\text{ N} \\ &= 28.66\text{ N} \approx 29\text{ N}. \qquad \text{(Answer)} \end{aligned}$$

12.6: Intermediate Structures

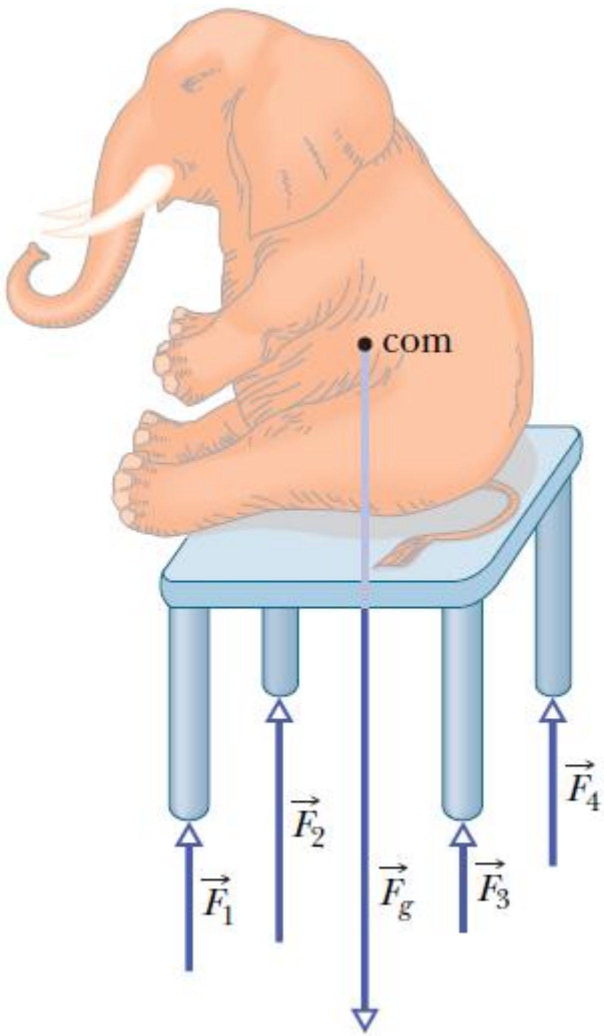


Fig. 12-8 The table is an indeterminate structure. The four forces on the table legs differ from one another in magnitude and cannot be found from the laws of static equilibrium alone.

12.7: Elasticity

- A **stress** is defined as deforming force per unit area, which produces a **strain**, or unit deformation.

- Stress and strain are proportional to each other. The constant of proportionality is called a **modulus of elasticity**.

Stress = Modulus x Strain

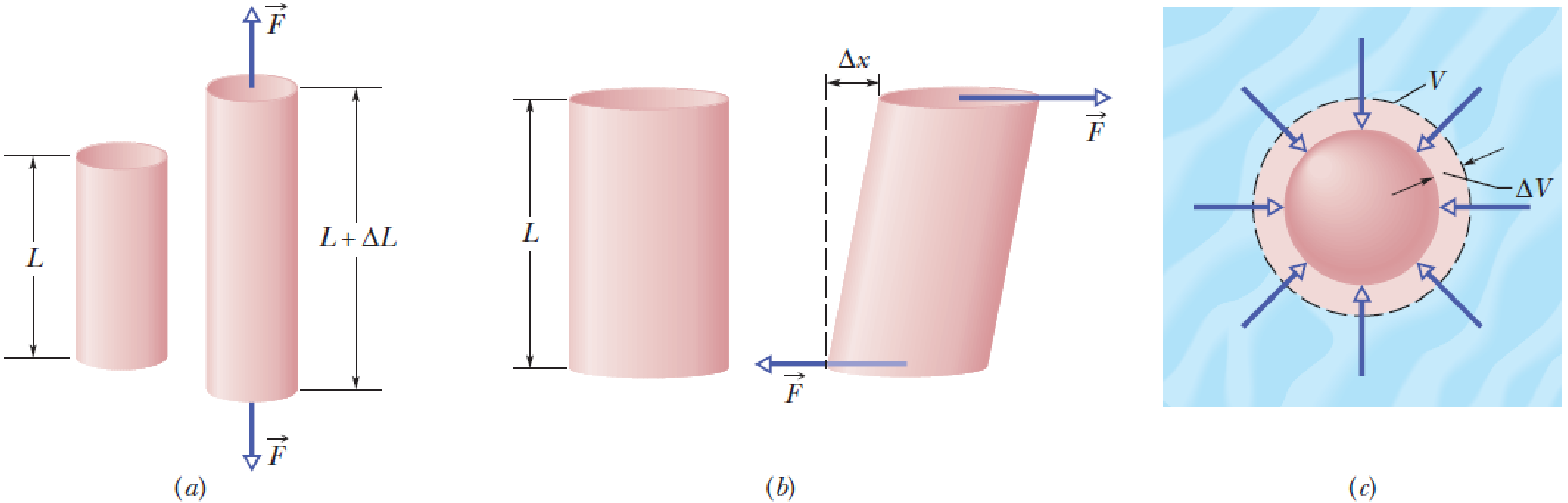


Fig. 12-10 (*a*) A cylinder subject to *tensile stress* stretches by an amount ΔL. (*b*) A cylinder subject to *shearing stress* deforms by an amount Δx, somewhat like a pack of playing cards would. (*c*) A solid sphere subject to uniform *hydraulic stress* from a fluid shrinks in volume by an amount ΔV. All the deformations shown are greatly exaggerated.

12.7 Elasticity: Tension and Compression

- For simple tension or compression, the stress on the object is defined as *F/A*, where F is the magnitude of the force applied perpendicularly to an area *A* on the object.

- The strain, or unit deformation, is then the dimensionless quantity *ΔL/L*, the fractional change in a length of the specimen.

- The modulus for tensile and compressive stresses is called the **Young's modulus** and is represented in engineering practice by the symbol *E.*

$$\frac{F}{A} = E\,\frac{\Delta L}{L}.$$

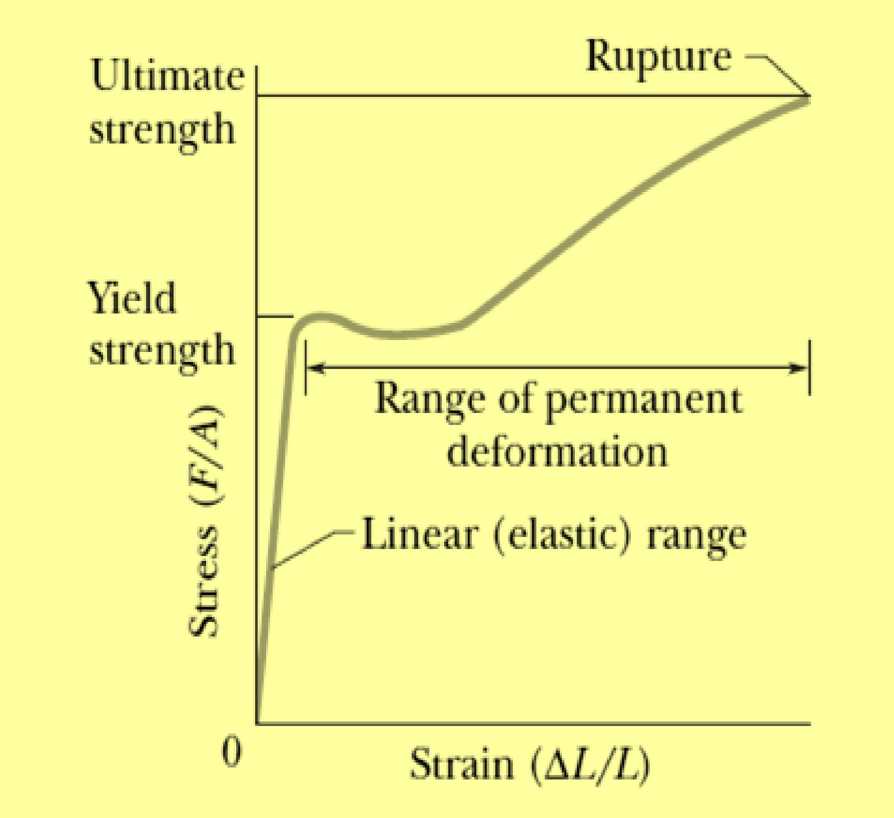


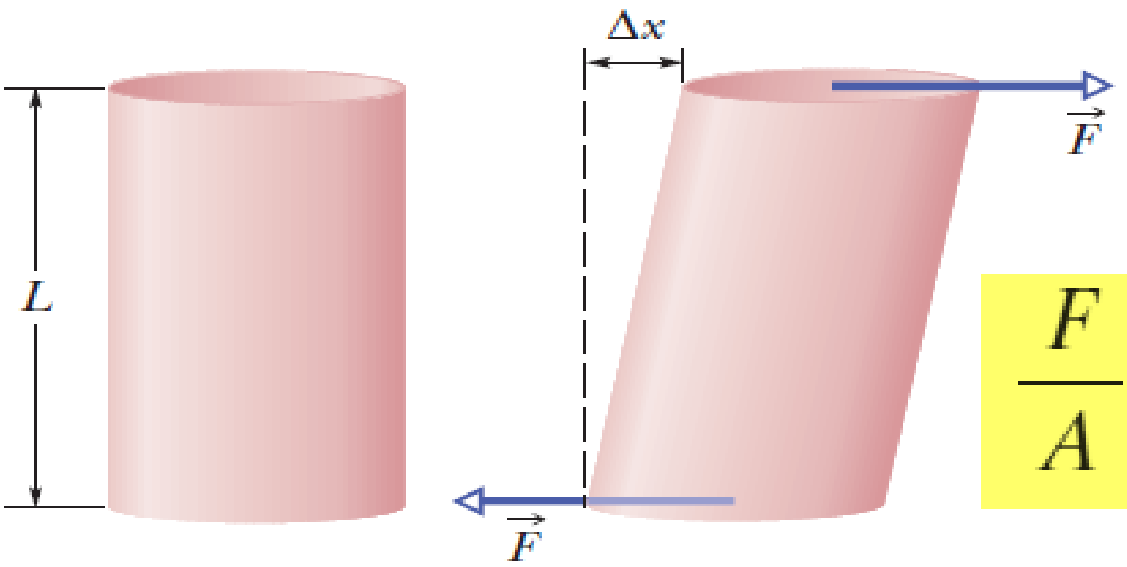
Fig. 12-12 *A stress–strain curve for a steel test specimen. The specimen deforms permanently when the stress is equal to the yield strength of the specimen's material. It ruptures when the stress is equal to the ultimate strength of the material.*

12.7: Elasticity: Shearing

- In the case of shearing, the stress is also a force per unit area, but the force vector lies in the plane of the area rather than perpendicular to it.

- The strain is the dimensionless ratio *Δx/L*, with the quantities defined as shown in the figure.

- The corresponding modulus, which is given the symbol *G* in engineering practice, is called the **shear modulus.**



$$\frac{F}{A} = G\,\frac{\Delta x}{L}.$$

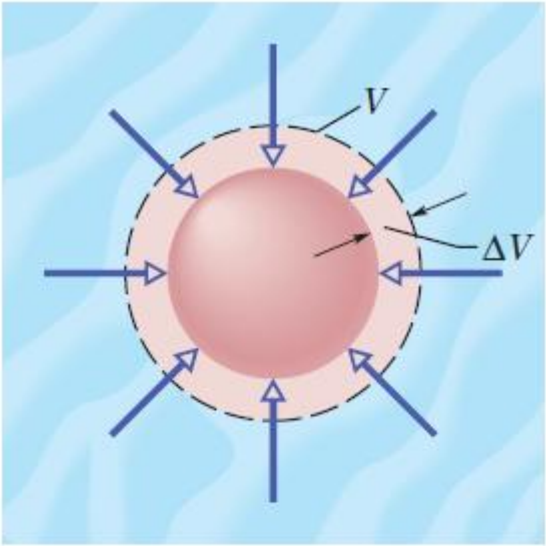
12.7 Elasticity: Hydraulic Stress

- In the figure, the stress is the fluid pressure p on the object, where pressure is a force per unit area.

- The strain is *ΔV/V*, where V is the original volume of the specimen and ΔV is the absolute value of the change in volume.

- The corresponding modulus, with symbol B, is called the **bulk modulus** of the material. The object is said to be under hydraulic compression, and the pressure can be called the hydraulic stress.

$$p = B\,\frac{\Delta V}{V}$$



12.7: Elasticity

Table 12-1

Some Elastic Properties of Selected Materials of Engineering Interest

Material	Density ρ (kg/m^3)	Young's Modulus E ($10^9\ N/m^2$)	Ultimate Strength S_u ($10^6\ N/m^2$)	Yield Strength S_y ($10^6\ N/m^2$)
Steel[a]	7860	200	400	250
Aluminum	2710	70	110	95
Glass	2190	65	50[b]	—
Concrete[c]	2320	30	40[b]	—
Wood[d]	525	13	50[b]	—
Bone	1900	9[b]	170[b]	—
Polystyrene	1050	3	48	—

[a]Structural steel (ASTM-A36).
[b]In compression.
[c]High strength
[d]Douglas fir.

Example: elongated rod

One end of a steel rod of radius $R = 9.5$ mm and length $L = 81$ cm is held in a vise. A force of magnitude $F = 62$ kN is then applied perpendicularly to the end face (uniformly across the area) at the other end, pulling directly away from the vise. What are the stress on the rod and the elongation ΔL and strain of the rod?

KEY IDEAS

(1) Because the force is perpendicular to the end face and uniform, the stress is the ratio of the magnitude F of the force to the area A. The ratio is the left side of Eq. 12-23. (2) The elongation ΔL is related to the stress and Young's modulus E by Eq. 12-23 ($F/A = E\,\Delta L/L$). (3) Strain is the ratio of the elongation to the initial length L.

Calculations: To find the stress, we write

$$\text{stress} = \frac{F}{A} = \frac{F}{\pi R^2} = \frac{6.2 \times 10^4\ \text{N}}{(\pi)(9.5 \times 10^{-3}\ \text{m})^2}$$
$$= 2.2 \times 10^8\ \text{N/m}^2. \qquad \text{(Answer)}$$

The yield strength for structural steel is 2.5×10^8 N/m^2, so this rod is dangerously close to its yield strength.

We find the value of Young's modulus for steel in Table 12-1. Then from Eq. 12-23 we find the elongation:

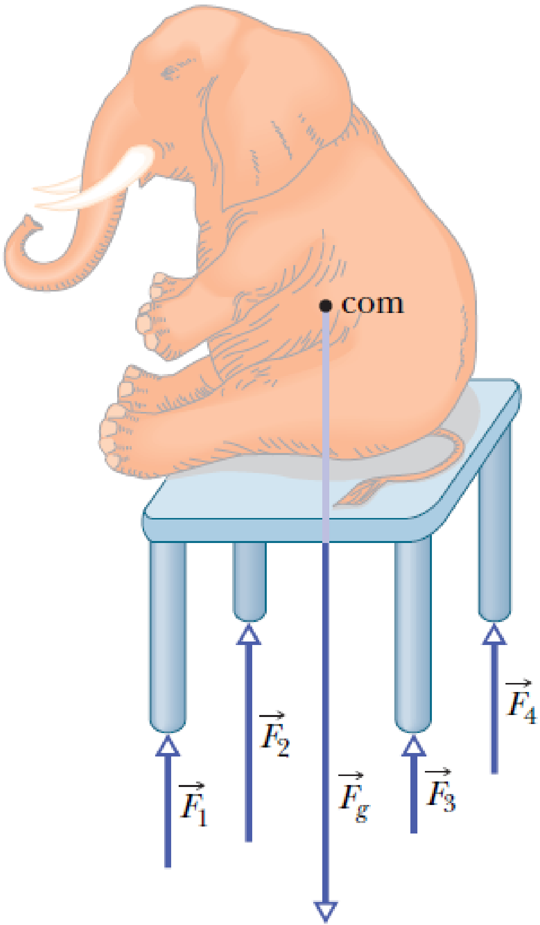
$$\Delta L = \frac{(F/A)L}{E} = \frac{(2.2 \times 10^8\ \text{N/m}^2)(0.81\ \text{m})}{2.0 \times 10^{11}\ \text{N/m}^2}$$
$$= 8.9 \times 10^{-4}\ \text{m} = 0.89\ \text{mm}. \qquad \text{(Answer)}$$

For the strain, we have

$$\frac{\Delta L}{L} = \frac{8.9 \times 10^{-4}\ \text{m}}{0.81\ \text{m}}$$
$$= 1.1 \times 10^{-3} = 0.11\%. \qquad \text{(Answer)}$$

Example: wobbly table

A table has three legs that are 1.00 m in length and a fourth leg that is longer by $d = 0.50$ mm, so that the table wobbles slightly. A steel cylinder with mass $M = 290$ kg is placed on the table (which has a mass much less than M) so that all four legs are compressed but unbuckled and the table is level but no longer wobbles. The legs are wooden cylinders with cross-sectional area $A = 1.0\ \text{cm}^2$; Young's modulus is $E = 1.3 \times 10^{10}\ \text{N/m}^2$. What are the magnitudes of the forces on the legs from the floor?



We take the table plus steel cylinder as our system. The situation is like that in the figure. If the tabletop remains level, the legs must be compressed in the following ways: Each of the short legs must be compressed by the same amount (call it ΔL_3) and thus by the same force of magnitude F_3. The single long leg must be compressed by a larger amount ΔL_4 and thus by a force with a larger magnitude F_4.

$$\Delta L_4 = \Delta L_3 + d. \longrightarrow \frac{F_4 L}{AE} = \frac{F_3 L}{AE} + d.$$

$$F_{\text{net},y} = 0 \longrightarrow 3F_3 + F_4 - Mg = 0,$$

$$\begin{aligned} F_3 &= \frac{Mg}{4} - \frac{dAE}{4L} \\ &= \frac{(290\ \text{kg})(9.8\ \text{m/s}^2)}{4} \\ &\quad - \frac{(5.0 \times 10^{-4}\ \text{m})(10^{-4}\ \text{m}^2)(1.3 \times 10^{10}\ \text{N/m}^2)}{(4)(1.00\ \text{m})} \\ &= 548\ \text{N} \approx 5.5 \times 10^2\ \text{N}. \qquad \text{(Answer)} \end{aligned}$$

$$\begin{aligned} F_4 &= Mg - 3F_3 = (290\ \text{kg})(9.8\ \text{m/s}^2) - 3(548\ \text{N}) \\ &\approx 1.2\ \text{kN}. \qquad \text{(Answer)} \end{aligned}$$